Mathematics of Computers

Created for Moreno Valley High School by Paul Ellsworth

Chapter 1 - Introduction to Number Systems

Objectives:
1 Students will know that every decimal numbers can be written as a sum of weighted exponents determined by each digit of a numbers position.

2 Students will know what the binary numbering system is and that every binary number can be written as a sum of weighted exponents determined by each digit of a numbers position.

CA Mathematics Algebra 1 Content Standard 2.0 – Students understand and use such operations as taking the opposite, finding the reciprocal, taking a root, and raising to a fractional power. They understand and use the rules of exponents.

CA Career Technical Education Model Curriculum Standard, Engineering Technology Pathway, D8.5 Understands the relationship among computer hardware, networks, and operating systems.

Section 1 - Early History of Computers

The first modern computer was built about sixty years ago. All computers share a common ancestor, the UNIVAC 1. This room-sized computer was built using vacuum tube technology in 1950 and has less computing power than the smallest of today's microcomputers.

The vacuum tube was a bistable device, which has two stable states, ON or OFF. Being large and bulky and prone to failure the vacuum tubes of that era were quickly replaced by discrete transistor which either conducted current or not. Today we still use the transistor as the main switching element in computers. Millions of transistors are tightly packed into each integrated circuit, which have two major effects: first, it reduced the cost of computers to a fraction of the UNIVAC 1 and second, an increase in performance by a factor of over a million. Virtually every home in America has at least one computer and it has become a critical component of modern society.

The numbering system that we use has ten digits, 0…9. We call our number system decimal or base 10. Making electronic devices with ten stable values would be extremely difficult and expensive while on the other hand making electronic circuits with two stable states is quite easy. Computers use only two digits, which we call binary or base 2 and computers simply process binary digits using Boolean Algebra. The binary digit has two values, 1 or 0.

This naturally brings up the question, "How can we represent decimal numbers using the binary numbering system?" In order to answer this question we have to understand how the decimal numbering system works.

If we start to count in decimal with the numbers 0, 1, 2, 3, 4, 5, 6, 7, 8, 9. The next numbers in the sequence are 10, 11, 12, and so on. Notice that the number 10 has two digits, a one and a zero. The one is in the ten's place while the zero is in the one's place. What it means is add one tens to zero ones or 10 + 0 = 10. Eleven means add one tens to one ones or 10 + 1 = 11. For a number like twenty-five it means add two tens to five ones or 20 + 5 = 25.

Let's consider a larger number like one hundred and twenty-three. Numerically we represent this as 123. What it tells us is to add one hundreds plus two tens plus three ones or 100 + 20 + 3 = 123.

In general, a number N can be represented in the following form:

$$N = d_{p-1} \times b^{p-1} + ... + d_{p-2} \times b^{p-2} + ... + d_0 \times b^0 + d_{-1} \times b^{-1} + ... + d_{-q} \times b^{-q}$$

where

b is the base or radix of the number system;
d's are the digits of the number system in which there are b allowable digits;
p is the number of integral digits;
q is the number of fractional digits;
and $(b–1) \geq d_i \geq 0$ where $(p-1) \geq i \geq (4–q)$

N can also be written as a string of digits whose integral and fractional parts are separated by a ".", which is called a radix point and takes on the name of the base or radix of the number system being utilized. In decimal it is called the decimal point and in binary it is called the binary point. When written N takes on the format as $d_{p-1}d_{p-2}...d_1d_0.d_{-1}...d_{-q}$. Notice the radix point is between d_0 and d_{-1}.

If a number does not have a factional portion the number is called an integer. Conversely, if the number has no integer portion the number is called a fraction. If there are both an integer portion and a fractional portion then the number is called a mixed number.

Since some digits may be common to more than one number system, the base or radix usually is written as a subscript following the number except in cases where it is specified beforehand or it is obvious. For example the decimal number 123.45 could be written as 123.45_{10}. Because we used the decimal number system everyday the subscript 10 is usually omitted.

Section 2 - Decimal Number System

In the decimal number system the number 123.45 is actually:

$$123.45 = 1\times10^2 + 2\times10^1 + 3\times10^0 + 4\times10^{-1} + 5\times10^{-2}$$

or

$$123.45 = 100 + 20 + 3 + \frac{4}{10} + \frac{5}{100}$$

We could also look at the number 123.45 in terms of weights corresponding to the position left or right of the decimal point occupied by a digit. For example, the 3 in the first position to the left of the decimal point is the units digit; the next digit to the left, 2, is the tens digit; the next digit to the left, 1, is in the hundreds position. We can use a table to represent the weighted number.

Weighted Representation of N = 123.45							
10,000	1,000	100	10	1	0.1	0.01	0.001
0	**0**	**1**	**2**	**3**	**4**	**5**	**0**

By virtue of their respective positions in the table, the digit 1 represents 1 x 100, the digit 2 represents 2 x 10, the digit 3 represents 3 x 1, the digit 4 represents 4 x 0.1, and the digit 5 represents 5 x 0.01. A zero in a column means that no contribution to the value N is made by the corresponding weight of that column. Therefore, the value of the resulting number, N, is the sum of the values contributed by each digit in its respective position. Additional columns could be extended to the left or right as needed with the weight of each additional column increasing or decreasing by a factor of ten from the preceding column.

Skills Review. Converting a fraction to a decimal number.

Convert the fraction $\frac{9}{20}$ to a decimal number.

You need to divide 20 into 9 $\Rightarrow$ $20\overline{)9.0}$

Perform the division – 20 goes into 90 four times

$$\begin{array}{r} .4 \\ 20\overline{)9.0} \\ -80 \\ \hline 10 \end{array}$$

Bring down another 0, divide 20 into 100. Twenty goes into 100 five times.

Our answer is: $\dfrac{9}{20} = 0.45$

Example 1. Convert the decimal number 5 to exponential form.

$$5 \times 10^0$$

Example 2. Convert 423_{10} to exponential form.

$$4 \times 10^2 + 2 \times 10^1 + 3 \times 10^0$$

Example 3. Convert 36.75_{10} to exponential form.

$$3 \times 10^1 + 6 \times 10^0 + 7 \times 10^{-1} + 5 \times 10^{-2}$$

Example 4. Convert the exponential number $9 \times 10^3 + 3 \times 10^1 + 4 \times 10^0 + 6 \times 10^{-1} + 8 \times 10^{-3}$ to a decimal number.

$$9 \times 10^3 = 9000$$
$$3 \times 10^1 = 30$$
$$4 \times 10^0 = 4$$
$$6 \times 10^{-1} = \frac{6}{10}$$
$$8 \times 10^{-3} = \frac{8}{1000}$$

Add the numbers up $\quad 9000 + 30 + 4 + \dfrac{6}{10} + \dfrac{8}{1000} =$

$$9000 + 30 + 4 + 0.6 + 0.008 = 9034.608$$

Section 3 - Binary Number System

The binary number system has a base or radix of two and has two allowable digits, 0 and 1. The binary number 101.01 can be written as 101.01_2 to indicate that each digit in the number corresponds to a power of two as opposed to a power of ten in the decimal number system. Thus 101.01_2 is interpreted as:

$$1 \times 2^2 + 0 \times 2^1 + 1 \times 2^0 + 0 \times 2^{-1} + 1 \times 2^{-2}$$

The decimal or base ten value of 101.01_2 is found from the above summation as:

$$4 + 0 + 1 + 0 + \frac{1}{4}$$

or

$$5.25_{10}$$

Looking at this binary number in a weighted representation with the weight of each position now being a power of two yields:

Weighted Representation of N = 101.01_2							
8	4	2	1	0.5	0.25	0.125	0.0625
0	**1**	**0**	**1**	**0**	**1**	**0**	**0**

Note that the value contributed by each binary digit to N is determined by its position relative to the binary point. The procedure for determining N is identical to that described for decimal numbers except that each position has a weight corresponding to a power of two instead of a power of ten.

The following table has the binary equivalents for the decimal numbers 0 through 15. It is important to become familiar with the sequence of binary numbers and also to note the pattern of alternating zeros and ones in each weight column of the binary number patterns.

Binary-Decimal Equivalent Numbers				
Binary Number Positional Weight				Decimal Number
8	4	2	1	
0	0	0	0	0
0	0	0	1	1
0	0	1	0	2
0	0	1	1	3
0	1	0	0	4
0	1	0	1	5
0	1	1	0	6
0	1	1	1	7
1	0	0	0	8
1	0	0	1	9
1	0	1	0	10
1	0	1	1	11
1	1	0	0	12
1	1	0	1	13
1	1	1	0	14
1	1	1	1	15

Example 5. Convert 1011_2 to exponential form.

$$1\times 2^3 + 1\times 2^1 + 1\times 2^0$$

Example 6. Convert 1100.011_2 to exponential form.

$$1\times 2^3 + 1\times 2^2 + 1\times 2^{-2} + 1\times 2^{-3}$$

Example 7. Convert $1\times 2^4 + 1\times 2^1 + 1\times 2^0 + 1\times 2^{-1} + 1\times 2^{-3}$ to a binary number.

$$2^4 \to 1$$
$$2^3 \to 0$$
$$2^2 \to 0$$
$$2^1 \to 1$$
$$2^0 \to 1$$
$$2^{-1} \to 1$$
$$2^{-2} \to 0$$
$$2^{-3} \to 1$$

$$\therefore 10011.101_2$$

Exercise 1

1. Convert the following decimal numbers to exponential form.

a. 4 b. 361 c. 2500 d. 0.3141 e. 744.35

2. Convert the following exponential numbers to decimal numbers.

a. 5×10^{0}
b. $1 \times 10^{2} + 6 \times 10^{1} + 3 \times 10^{0}$
c. $2 \times 10^{-1} + 5 \times 10^{-2}$
d. $1 \times 10^{0} + 3 \times 10^{-1} + 0 \times 10^{-2} + 7 \times 10^{-3}$
e. $4 \times 10^{3} + 1 \times 10^{2} + 5 \times 10^{0} + 9 \times 10^{-1} + 2 \times 10^{-2}$

3. Convert the following binary numbers to exponential form.

a. 10
b. 11
c. 1101
d. 1001.11
e. 11011.01011

4. Convert the following exponential numbers to binary numbers.

a. $1 \times 2^{1} + 0 \times 2^{0}$
b. $1 \times 2^{2} + 1 \times 2^{1} + 0 \times 2^{0}$
c. $1 \times 2^{-1} + 0 \times 2^{-2} + 0 \times 2^{-3} + 1 \times 2^{-4}$
d. $1 \times 2^{6} + 1 \times 2^{4} + 1 \times 2^{3} + 1 \times 2^{0}$
e. $1 \times 2^{4} + 1 \times 2^{3} + 1 \times 2^{1} + 1 \times 2^{0} + 1 \times 2^{-1} + 1 \times 2^{-4}$

5. Define the term radix.

6.Why is the binary number system used in computers?

7. What is the decimal value of the largest number that can be represented by a 10-digit binary number?

8. How many binary digits would it take to represent the decimal number 1000?